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Cartan-Kähler Theory and Applicationsto Local Isometric and Conformal Embedding

Nabil Kahouadji

Abstract

The goal of this lecture is to give a brief introduction to Cartan-Kähler's theory. As examples to the application of this theory, we choose the local isometric and conformal embedding. We provide lots of details and explanations of the calculation and the tools used¹.

1 Cartan's Structure Equations

Let $\xi = (E, \pi, M)$ be a vector bundle. Denote $(\mathfrak{X}(M), [,])$ the Lie algebra of vector fields on M and $\Gamma(E)$ the moduli space of cross-sections of the vector bundle E .

1.1 Connection on a vector bundle

A connection on a vector bundle E is a choice of complement of vertical vector fields on E . A connection induces a covariant differential operator ∇ on E . A covariant derivative ∇ on a vector bundle E is a way to "differentiate" bundle sections along tangent vectors and it is sometimes called a connection.

Definition 1.1.1. *A connection on a vector bundle E is an linear operator defined as follows:*

$$\begin{aligned}\nabla : \mathfrak{X}(M) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ (X, S) &\longmapsto \nabla_X S\end{aligned}$$

¹See the Master Thesis [7] on which the lecture is based

satisfying

$$\begin{aligned}\nabla_{(X_1+X_2)}S &= \nabla_{X_1}S + \nabla_{X_2}S, & \nabla_{(fX)}S &= f\nabla_XS \\ \nabla_X(S_1 + S_2) &= \nabla_XS_1 + \nabla_XS_2, & \nabla_X(fS) &= X(f)S + f\nabla_XS \\ \forall X, X_1, X_2, Y, Y_1, Y_2 &\in \mathfrak{X}(M) \text{ and } \forall S, S_1, S_2 \in \Gamma(E).\end{aligned}$$

1.1.1 Curvature of a Connection

Definition 1.1.2. The curvature of a connection ∇ is a vector valued 2-form

$$\begin{aligned}\mathcal{R} : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ X, Y, S &\longmapsto \mathcal{R}(X, Y)S\end{aligned}$$

defined by $\mathcal{R}(X, Y)S = ([\nabla_X, \nabla_Y] - \nabla_{[X, Y]})S$

Theorem 1.1. For any f, g and h smooth functions on M , $S \in \Gamma(E)$ a section of ξ and $X, Y \in \mathfrak{X}(M)$ two tangent vector fields of M , we have

$$\mathcal{R}(fX, gY)(hS) = fgh\mathcal{R}(X, Y)S \quad (1)$$

1.1.2 Connection and Curvature Forms

Let $\xi = (E, \pi, M)$ be a vector bundle over a smooth manifold M with an r -dimensional vector space E as a standard fiber. Let ∇ be a connection on ξ and \mathcal{R} its curvature. We denote by \mathcal{U} an open set of M .

Definition 1.1.3. A set of r local sections $S = (S_1, S_2, \dots, S_r)$ of ξ is called a frame field (or a moving frame) if $\forall p \in \mathcal{U}$, $S(p) = (S_1(p), S_2(p), \dots, S_r(p))$ form a basis of the fiber E_p over p .

Let $S = (S_1, S_2, \dots, S_r)$ be a frame field, ∇ a connection on ξ and $X \in \mathfrak{X}(M)$ a tangent vector field on M . Then $\nabla_X S_j$ is another section of ξ and it can be expressed in the frame field S as follows:

$$\nabla_X S_j = \sum_{i=1}^r \omega_{ij}(X) S_i \quad (2)$$

where $\omega_{ij} \in \mathcal{A}^1(M)$ are differential 1-forms² on M and $\omega_{ij}(X)$ are smooth functions on M .

² $\mathcal{A}^k(M)$ denote the set of differential k -form on M (we choose this notation instead of the standard notation $\Omega^k(M)$ to not mix with the curvature form).

Definition 1.1.4. The $r \times r$ matrix $\omega = (\omega_{ij})$ is called the connection 1-form of ∇ .

The connection ∇ is completely determined by the matrix $\omega = (\omega_{ij})$. Conversely, a matrix of differential 1-forms on M determines a connection (in a non-invariant way depending on the choice of the moving frame).

Let $X, Y \in \mathfrak{X}(M)$ two tangent vector fields. Then $\mathcal{R}(X, Y)S_j$ are sections of ξ , and can be expressed on the frame field S as follows:

$$\mathcal{R}(X, Y)S_j = \sum_{i=1}^r \Omega_{ij}(X, Y)S_i \quad (3)$$

where $\Omega_{ij} \in \mathcal{A}^2(M)$ are differential 2-forms on M and $\Omega_{ij}(X, Y)$ are smooth functions on M .

Definition 1.1.5. The $r \times r$ matrix $\Omega = (\Omega_{ij})$ whose entries are differential 2-forms, is called the curvature 2-form of the connection ∇ .

We state the following theorem³ that gives the relation between the connection 1-form ω and the curvature 2-form Ω .

Theorem 1.2.

$$d\omega + \omega \wedge \omega = \Omega \quad (\text{matrix form}) \quad (4)$$

or

$$d\omega_{ij} + \sum_{k=1}^m \omega_{ik} \wedge \omega_{kj} = \Omega_{ij} \quad (\text{on components}) \quad (5)$$

1.2 The Induced Connection

Let $\xi = (E, \pi, M)$ and $\xi' = (E', \pi', M)$ be two vector bundles on M . Consider a map $f : M \rightarrow M$ and denote $\tilde{f} : E \rightarrow E'$ the associated bundle map i.e. (f, \tilde{f}) satisfies the following commutative diagramme:

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{f} & M \end{array}$$

³In the tangent bundle case, this theorem gives Cartan's second equation, as we will see later.

If ∇' is a connection on E' , the vector bundle morphism induces a pull-back connection on E

$$\nabla = \tilde{f}^* \nabla' \quad (6)$$

such that for any $S' \in \Gamma(E')$ and $X \in \mathfrak{X}(M)$, $\nabla_X(\tilde{f}^* S') = (\tilde{f}^* \nabla')_X(\tilde{f}^* S') = \tilde{f}^* (\nabla_{f_* X} S')$ where $f_{*,p} : T_p M \rightarrow T_{f(p)} M$ is the linear tangent map.

We can also induce a connection on ξ by another way. The connection ∇' is completely determined by the matrix of differential 1-forms $\omega' = (\omega'_{ij})$, and we define ∇ by the matrix ω whose entries ω_{ij} are the pull-back of ω'_{ij} by \tilde{f} , i.e. $\omega = \tilde{f}^* \omega'$.

The pull back commute with the exterior differentiation and with the exterior product⁴, so, the curvature 2-form Ω of ∇ is the pull back of the curvature 2-form of ∇' , i.e. $\Omega = \tilde{f}^* \Omega'$.

1.3 Metric Connection

Let $\xi = (E, \pi, M)$ be a vector bundle. We denote by ∇ a connection on ξ determined by a matrix of 1-forms ω . Let Ω be the associated curvature 2-form and g a Riemannian metric on ξ (i.e. a positively-defined scalar product on each fiber).

Definition 1.3.1. ∇ is a connection on ξ compatible with the metric g (or a metric connection) if ∇ satisfies to the following property (Leibniz's identity):

$$\begin{aligned} \nabla_X (g(S_1, S_2)) &= g(\nabla_X S_1, S_2) + g(S_1, \nabla_X S_2) \\ \forall S_1, S_2 \in \Gamma(E), \text{ and } \forall X \in \mathfrak{X}(M) \end{aligned} \quad (7)$$

Proposition 1.3.1. Let $S = (S_1, S_2, \dots, S_n)$ be an orthonormal frame field with respect to g , i.e. $g_p(S_i, S_j) = \delta_{ij}$ for all $p \in \mathcal{U}$, $i, j = 1, \dots, r$, then the matrix of 1-forms ω associated to S and the curvature matrix of 2-form are both skew-symmetric.

⁴ $d(f^* \alpha) = f^*(d\alpha)$ and $f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta)$ for all $\alpha, \beta \in \mathcal{A}(M)$.

1.4 Tangent Bundle Case

1.4.1 Torsion of a Connexion on a Tangent Bundle

Let us consider now, a local frame field $S = (S_1, \dots, S_m)$ over $\mathcal{U} \subset M$ where $S_i \in \mathfrak{X}(\mathcal{U})$ are local tangent vectors fields (i.e. local sections of the tangent bundle such that for all $p \in \mathcal{U}$, $(S_1(p), \dots, S_m(p))$ forms a basis of the tangent vector space of M).

Definition 1.4.1. *If S is a local orthonormal frame field, the associated coframe field $\eta = (\eta_1, \dots, \eta_m)$ is a local frame field of 1-forms, such that for all $p \in \mathcal{U}$, $\eta_i(p)(S_j) = \delta_{ij}$.*

We define then a differential 2-form Θ as follows:

$$d\eta + \omega \wedge \eta = \Theta \quad (8)$$

Definition 1.4.2. Θ is called the torsion 2-form of ∇ .

Proposition 1.4.1. *On a tangent bundle, the four forms η, ω, Θ and Ω are connected by the following equations*

$$d\Theta + \omega \wedge \Theta = \Omega \wedge \eta \quad (9)$$

and

$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega \quad (10)$$

The equation (10) is the expression of the Bianchi identity via the connection 1-form and the curvature 2-form. Equation (10) is also valid on a arbitrary vector bundle.

1.4.2 Cartan's Structure Equations

Let (M, g) be an m -dimensional Riemannian manifold. Let $\eta = (\eta_1, \eta_2, \dots, \eta_m)$ an orthonormal coframe field on M ($\eta_j \in \mathcal{A}^1(M)$). According to equations (8), (5) and the proposition 1.3.1, we establish the Cartan structure equations:

$$\begin{cases} d\eta_i + \sum_{j=1}^m \omega_{ij} \wedge \eta_j = 0 & \text{(torsion-free)} \\ d\omega_{ij} + \sum_{k=1}^m \omega_{ik} \wedge \omega_{kj} = \Omega_{ij} \end{cases} \quad (11)$$

where the matrix (ω_{ij}) is the Lévi-Civita connection 1-form (free torsion connection which is compatible with the riemannian metric g). Because η is an orthonormed coframe field, (ω_{ij}) is skew-symmetric (proposition 1.3.1.). (Ω_{ij}) is the curvature 2-form matrix of the riemannian connection $(\Omega_{ij} = \frac{1}{2} \sum_{k,l=1}^m \mathcal{R}_{ijkl} \eta_k \wedge \eta_l)$.

1.5 The Cartan Lemma

We end this section with a technical lemma, which is easy to establish and at the same time rich applications. This lemma will not only be useful for isometric embedding problem, but also for many calculus in differential geometry.

Lemma 1.5.1. *Let M be an m -dimensional manifold. $\omega_1, \omega_2, \dots, \omega_r$ a set of linearly independent differential 1-forms ($r \leq m$) and $\theta_1, \theta_2, \dots, \theta_r$ differential 1-forms such that*

$$\sum_{i=1}^r \theta_i \wedge \omega_i = 0 \quad (12)$$

then there exists r^2 functions h_{ij} in $\mathcal{C}^1(M)$ such that

$$\theta_i = \sum_{j=1}^r h_{ij} \omega_j \quad \text{with } h_{ij} = h_{ji}. \quad (13)$$

2 Exterior Differential Systems and Ideals

2.1 Exterior Differential Systems

Denote $\mathcal{A}(M)$ the space of smooth differential forms⁵ on M .

Definition 2.1.1. *An exterior differential system is a finite set of differential forms $I = \{\omega_1, \omega_2, \dots, \omega_k\} \subset \mathcal{A}(M)$ for which there is a set of equations $\{\omega_i = 0 | \omega_i \in I\}$.*

such that one can write the exterior differential system as follow:

$$\begin{cases} \omega_1 = 0 \\ \omega_2 = 0 \\ \vdots \\ \omega_k = 0 \end{cases}$$

⁵This is a graded algebra under the wedge product.

Definition 2.1.2. An exterior differential system $I \subset \mathcal{A}(M)$ is said to be Pfaffian if I contains only differential 1-forms, i.e. $I \subset \mathcal{A}^1(M)$.

2.2 Exterior Ideals

Definition 2.2.1. Let $\mathcal{I} \subset \mathcal{A}(M)$ a set of differentiable forms. \mathcal{I} is an exterior ideal if:

1. The exterior product of any differential form of \mathcal{I} by a differential form of $\mathcal{A}(M)$ belong to \mathcal{I} .
2. The sum of any two differential forms of the same degree belonging to \mathcal{I} , belong also to \mathcal{I} .

Definition 2.2.2. Let $I \subset \mathcal{A}(M)$ an exterior differential system. The exterior ideal generated by I is the smallest exterior ideal containing I .

2.3 Exterior Differential Ideals

Definition 2.3.1. Let $\mathcal{I} \subset \mathcal{A}(M)$ a set of differential forms. \mathcal{I} is an exterior differential ideal if \mathcal{I} is an exterior ideal closed under the exterior differentiation, i.e. $\forall \omega \in \mathcal{I}, d\omega \in \mathcal{I}$ (we can also write $d\mathcal{I} \subset \mathcal{I}$).

Definition 2.3.2. Let $I \subset \mathcal{A}(M)$ an exterior differential system. The exterior differential ideal generated by I is the smallest exterior differential ideal containing I .

2.4 Closed Exterior Differential Systems

Definition 2.4.1. An exterior differential system $I \subset \mathcal{A}(M)$ is said closed if the exterior differentiation of any form of I , belong to the exterior ideal generated by I .

Proposition 2.4.1. An exterior differential system I is closed if and only if the exterior differential ideal generated by I is equal to the exterior ideal generated by I . In particular, $I \cup dI$ is closed.

2.5 Solutions of an Exterior Differential System

Definition 2.5.1. Let $I \subset \mathcal{A}(M)$ be an exterior differential system and N a sub-manifold of M . N is an integral manifold of I if $i^*\omega = 0, \forall \omega \in I$, where i is an embedding $i : N \rightarrow M$.

3 Introduction to Cartan-Kähler Theory

We consider in this section, an m -dimensional real manifold M and $\mathcal{I} \subset \mathcal{A}(M)$ an exterior differential ideal on M .

3.1 Integral Elements

Definition 3.1.1. Let $z \in M$ and $E \subset T_z M$ a linear subspace of $T_z M$. E is an integral element of \mathcal{I} if $\varphi_E = 0$ for all $\varphi \in \mathcal{I}$. We denote by $\mathcal{V}_p(\mathcal{I})$ the set of p -dimensional integral elements of \mathcal{I} .

Definition 3.1.2. N is an integral manifold of \mathcal{I} if and only if each tangent space of N is an integral element of \mathcal{I} .

Proposition 3.1.1. If E is a p -dimensional integral element of \mathcal{I} , then every subspace of E are also integral elements of \mathcal{I} .

We denote by $\mathcal{I}_p = \mathcal{I} \cap \mathcal{A}^p(M)$ the set of differential p -forms of \mathcal{I} .

Proposition 3.1.2. $\mathcal{V}_p(\mathcal{I}) = \{E \in G_p(TM) \mid \varphi_E = 0 \text{ for all } \varphi \in \mathcal{I}_p\}$

Definition 3.1.3. Let E an integral element of \mathcal{I} . Let $\{e_1, e_2, \dots, e_p\}$ a basis of $E \subset T_z M$. The polar space of E , denoted by $H(E)$, is the vector space defined as follow:

$$H(E) = \{v \in T_z M \mid \varphi(v, e_1, e_2, \dots, e_p) = 0 \text{ for all } \varphi \in \mathcal{I}_{p+1}\}. \quad (14)$$

Notice that $E \subset H(E)$. This implies that a differential form is alternate. The polar space plays an important role in exterior differential system theory as we shall see in the following proposition.

Proposition 3.1.3. Let $E \in \mathcal{V}_p(\mathcal{I})$ be an p -dimensional integral element of \mathcal{I} . A $(p+1)$ -dimensional vector space $E^+ \subset T_z M$ which contains E is an integral element of \mathcal{I} if and only if $E^+ \subset H(E)$.

In order to check if a given p -dimensional integral element of an exterior differential ideal \mathcal{I} is contained in a $(p+1)$ -dimensional integral element of \mathcal{I} , we introduce the following function $r : \mathcal{V}_p(\mathcal{I}) \rightarrow \mathbb{Z}$, $r(E) = \dim H(E) - (p+1)$ is a relative integer, $\forall E \in \mathcal{V}_p(\mathcal{I})$.

Notice that $r(E) \geq 1$. If $r(E) = -1$, then E is contained in any $(p+1)$ -dimensional integral element of \mathcal{I} .

3.1.1 Kähler-Ordinary and Kähler-Regular Integral Elements

Let Δ a differential n -form on a m -dimensional manifold M . Let⁶ $G_n(TM, \Delta) = \{E \in G_n(TM) / \Delta_E \neq 0\}$. We denote by $\mathcal{V}_n(\mathcal{I}, \Delta) = \mathcal{V}_n(\mathcal{I}) \cap G_n(TM, \Delta)$ the set of integral elements of \mathcal{I} on which $\Delta_E \neq 0$.

Definition 3.1.4. *An integral element $E \in \mathcal{V}_n(\mathcal{I})$ is called Kähler-ordinary if there exists a differential n -form Δ such that $\Delta_E \neq 0$. Moreover, if the function r is locally constant in some neighborhood of E , then E is said Kähler-regular.*

3.1.2 Integral Flags, Ordinary and Regular Integral Elements

Definition 3.1.5. *An integral flag of \mathcal{I} on $z \in M$ of length n is a sequence of integral elements E_k of \mathcal{I} : $(0)_z \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset T_z M$.*

Definition 3.1.6. *Let I be an exterior differential system on M . An integral element $E \in \mathcal{V}(I)$ is said ordinary if its base point $z \in M$ is an ordinary zero of $I_0 = I \cap \mathcal{A}^0(M)$ and if there exists an integral flag $(0)_z \subset E_1 \subset E_2 \subset \cdots \subset E_n = E \subset T_z M$ where the E_k , $k = 1, \dots, (n-1)$ are Kähler-regular integral elements. Moreover, if E is Kähler-regular, then E is said regular.*

3.2 Cartan's Test

Theorem 3.1. *(Cartan's test)*

Let $\mathcal{I} \subset \mathcal{A}^(M)$ be an exterior ideal which does not contain 0-forms (functions on M). Let $(0)_z \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset T_z M$ be an integral flag of \mathcal{I} . For any $k < n$, we denote by c_k the codimension of the polar space $H(E_k)$ in $T_z M$. Then $\mathcal{V}_n(\mathcal{I}) \subset G_n(TM)$ is at least of $c_0 + c_1 + \cdots + c_{n-1}$ codimension at E_n . Moreover, E_n is an ordinary integral flag if and only if E_n has a neighborhood U in $G_n(TM)$ such that $\mathcal{V}_n(\mathcal{I}) \cap U$ is a manifold of $c_0 + c_1 + \cdots + c_{n-1}$ codimension in U .*

Proof. See [1], page 74. □

Proposition 3.2.1. *Let $\mathcal{I} \subset \mathcal{A}^*(M)$ an exterior ideal which do not contains 0-forms. Let $E \in \mathcal{V}_n(\mathcal{I})$ be an integral element of \mathcal{I} at the point $z \in M$. Let $\omega_1, \omega_2, \dots, \omega_n, \pi_1, \pi_2, \dots, \pi_s$ (where $s = \dim M - n$) be a coframe in a open neighborhood of $z \in M$ such that $E = \{v \in T_z M / \pi_a(v) = 0 \text{ for all } a = 1, \dots, s\}$. For all $p \leq n$, we define $E_p = \{v \in E / \omega_k(v) = 0 \text{ for all } k > p\}$. Let $\{\varphi_1, \varphi_2, \dots, \varphi_r\}$ be the set of differential forms which generate the exterior*

⁶ $G_n(TM)$ is the Grassmanian of TM , i.e. the set of n -dimensional subspace of TM .

ideal \mathcal{I} , where φ_ρ is of $(d_\rho + 1)$ degree.
For all ρ , there exists an expansion

$$\varphi_\rho = \sum_{|J|=d_\rho} \pi_\rho^J \wedge \omega_J + \tilde{\varphi}_\rho \quad (15)$$

where the 1-forms π_ρ^J are linear combinations of the forms π and the terms $\tilde{\varphi}_\rho$ are, either of degree 2 or more on π , or vanish at z .
moreover, we have

$$H(E_p) = \{v \in T_z M \mid \pi_\rho^J(v) = 0 \text{ for all } \rho \text{ and } \sup J \leq p\} \quad (16)$$

In particular, for the integral flag $(0)_z \subset E_1 \subset E_2 \subset \cdots \subset E_n \cap T_z M$ de \mathcal{I} , c_p correspond to the number of linear independent forms $\{\pi_\rho^J|_z \text{ such that } \sup J \leq p\}$.

Proof. See [1], page 80. □

3.3 Cartan-Kähler's Theorem

The following theorem is a generalization of the well-known Frobenius's theorem.

Theorem 3.2. (*Cartan-Kähler*)

Let $\mathcal{I} \subset \mathcal{A}^*(M)$ be a real analytic exterior differential ideal. Let $P \subset M$ a p -dimensional connected real analytic Kähler-Regular integral manifold of \mathcal{I} . Suppose that $r = r(P) \geq 0$. Let $R \subset M$ be a real analytic submanifold of M of codimension r which contains P and such that $T_x R$ and $H(T_x P)$ are transversals in $T_x M$ for all $x \in P \subset M$.

There exists a $(p + 1)$ -dimensional connected real analytic integral manifold X of \mathcal{I} , such that $P \subset X \subset R$. X is unique in the sense that another integral manifold of \mathcal{I} having the stated properties, coincides with X on a open neighborhood of P .

Proof. See [1], page 82. □

The analicity condition of the exterior differential ideal is crucial because of the requirements in the Cauchy-Kowalewski theorem used in the proof of the Cartan-Kähler theorem.

Cartan-Kähler's theorem has an important corollary. Actually, this corollary is often more used than the theorem and it is sometimes called *the Cartan-Kähler theorem*.

Corollary 3.3.1. (*Cartan-Kähler*)

Let \mathcal{I} be an analytic exterior differential ideal on a manifold M . If $E \subset T_z M$ is an ordinary integral element of \mathcal{I} , there exists an integral manifold of \mathcal{I} passing through z and having E as a tangent space at the point z .

4 Local Isometric Embedding Problem

We shall state and prove the Burstin-Cartan-Janet-Schlaefli's theorem concerning local isometric embedding of a real analytic Riemannian manifold. The names of the mathematicians are given in alphabetic order. Schlaefli in his paper in 1871 [8] conjectured that an m -dimensional Riemannian manifold can always be, locally, embedded in an $N = \frac{1}{2}m(m+1)$ dimensional Euclidean space. In 1926, Janet [6] proved the result for the dimension 2 by resolving a differential system and explain how we get the result in the general case. In 1927, Élie Cartan [3] gave the complete proof of the result. His method is based on his theory of involutive Pfaffian system. Later in 1931, Burstin [2] generalized Janet's method and obtained the result in the general case.

The proof that we shall give is inspired by Cartan's paper [3], the Bryant, Chern, Gardner, Goldschmidt et Griffiths's book [1] and the Griffiths et Jensen's book [4].

4.1 The Burstin-Cartan-Janet-Schlaefli theorem

Theorem 4.1. (*Burstin 1931-Cartan 1927-Janet 1926-Schlaefli 1871*)

Every m -dimensional real analytic Riemannian manifold can be locally embedded isometrically in an $\frac{m(m+1)}{2}$ -dimensional Euclidean space.

4.2 Proof of Burstin-Cartan-Janet-Schlaefli's theorem

Steps of the proof of theorem 4.1.

1. We shall write down the Cartan structure equations for an m -dimensional real analytic Riemannian manifold M .
2. We shall define a subbundle $\mathcal{F}_m(\mathbb{E}^N)$ of the bundle $\mathcal{F}(\mathbb{E}^N)$ of the Euclidean space \mathbb{E}^N , then shall write down the Cartan structure equations for the subbundle $\mathcal{F}_m(\mathbb{E}^N)$.
3. Given an exterior differential system I_0 on $M \times \mathcal{F}_m(\mathbb{E}^N)$, which is not close, we shall prove Claim 4.2.2, which proves that the existence of a

local isometric embedding of M is the existence of an m -dimensional integral manifold of I_0 .

4. We will extend this differential system to obtain a closed one. In the process of extension, we will get new equations (the Gauss equation (equ. 29)). We will also show that a closed exterior differential system \tilde{I} with fewer 1-forms than I , will generate the same differential ideal that the one generated by I if the Gauss's equation is satisfied.
5. We establish the lemma 4.2.1., that ensure that the Gauss equations is a surjective submersion. We shall obtain a submanifold with a known dimension.
6. Given the closed exterior differential ideal,, we shall prove the existence of an ordinary integral element by using claim 3.2.1 and the Cartan test. Finally, the Cartan-Kahler theorem ensure then the existence of an integral manifold and lead us to conclude..

Step 1:

Let (M, g) be an m -dimensional real analytic Riemannian manifold, where g is a Riemannian metric, i.e. a covariant symmetric positive defined 2-tensor, such that at a given point p of M , g_p in a orthonormed basis reduce to the identity matrix. However in a open neighborhood of p , the matrix of g can not always be the identity but it can always be reduced to diagonal matrix:

$$g = g_{11}dx^1 \otimes dx^1 + g_{22}dx^2 \otimes dx^2 + \cdots + g_{mm}dx^m \otimes dx^m \quad (17)$$

where the terms g_{ii} are positive functions such that $g_{ii} = 1$ at p . We denote than $\eta_i = \sqrt{g_{ii}}dx^i$. g can be written as follows:

$$g = \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \cdots + \eta_m \otimes \eta_m \quad (18)$$

$\eta = (\eta_1, \eta_2, \dots, \eta_m)$ is than a orthonormal coframe in the neighborhood of $p \in M$. We can establish the Cartan's structure equations:

Cartan's structure equations on M :

$$\begin{cases} d\eta_i + \sum_{j=1}^m \eta_{ij} \wedge \eta_j = 0 & (\text{torsion-free}) \\ d\eta_{ij} + \sum_{k=1}^m \eta_{ik} \wedge \eta_{kj} = \Omega_{ij} \end{cases} \quad (19)$$

where (η_{ij}) is the matrix of 1-form of the Lévi-Civita's connection on M (a torsion-free connection compatible with the metric g). Ω_{ij} is the curvature 2-form of the connection.

Step 2:

Let \mathbb{E}^N be an N -dimensional Euclidean space (for the moment, $N > m$) endowed with the usual scalar product ε_N . Let us consider $\mathcal{F}(\mathbb{E}^N)$ a positively-oriented orthonormal frame bundle on \mathbb{E}^N . In what follows, we will not work on the entire bundle $\mathcal{F}(\mathbb{E}^N)$, but on a quotient. An element in $\mathcal{F}_m(\mathbb{E}^N)$ has the form $(x; e_1, e_2, \dots, e_m)$, where $x \in \mathbb{E}^N$ and (e_1, e_2, \dots, e_m) is a positively-oriented orthonormal set of vectors in \mathbb{E}^N . We can consider $\mathcal{F}_m(\mathbb{E}^N)$ as follows: among all the positively-oriented orthonormal frames of $\mathcal{F}(\mathbb{E}^N)$, we take the frames such that the first m elements form a positively-oriented orthonormal set of vectors, then we take the m first vectors of these frames. So, $\mathcal{F}_m(\mathbb{E}^N)$ is diffeomorphic to $\mathbb{E}^N \times \frac{SO(N)}{SO(N-m)}$.

Proposition 4.2.1.

$$\dim \mathcal{F}_m(\mathbb{E}^N) = N(m+1) - \frac{m(m+1)}{2} \quad (20)$$

We define on $\mathcal{F}(\mathbb{E}^N)$ a set of 1-forms as follows⁷:

$$\omega_\mu = e_\mu dx \quad \text{and} \quad \omega_{\mu\nu} = e_\mu de_\nu - e_\nu de_\mu = -\omega_{\nu\mu} \quad (21)$$

So $(\omega_1, \omega_2, \dots, \omega_m, \omega_{m+1}, \dots, \omega_N)$ form an orthonormal coframe of $\mathcal{F}(\mathbb{E}^N)$. Then the Cartan structure equations on $\mathcal{F}_m(\mathbb{E}^N)$ are:

$$\begin{cases} d\omega_\mu + \sum_{\nu=1}^N \omega_{\mu\nu} \wedge \omega_\nu = 0 & (\text{torsion-free}) \\ d\omega_{\mu\nu} + \sum_{\lambda=1}^N \omega_{\mu\lambda} \wedge \omega_{\lambda\nu} = 0 & (\text{flat curvature}) \end{cases} \quad (22)$$

Notice that $(\omega_{\mu\nu})$ is the $N \times N$ skew-symmetric matrix connection form of the Lévi-Civita connection on \mathbb{E}^N .

⁷The indices i, j and k vary from 1 to m , the indexes a, b and c vary from $m+1$ to N and the indexes μ, ν and λ vary from 1 to N .

Step 3:

Let consider the product manifold $M \times \mathcal{F}_m(\mathbb{E}^N)$. Let \mathcal{I}_0 be the exterior ideal on $M \times \mathcal{F}_m(\mathbb{E}^N)$ generated by the Paffafian system $I_0 = \{(\omega_i - \eta_i), \omega_a\}$.

Proposition 4.2.2. *Every m -dimensional integral manifold of \mathcal{I}_0 on which the form $\Delta = \omega_1 \wedge \cdots \wedge \omega_m$ does not vanish is locally the graph of a function $f : M \longrightarrow \mathcal{F}_m(\mathbb{E}^N)$ having the property that $u = x \circ f : M \longrightarrow \mathbb{E}^N$ is a local isometric embedding⁸.*

$$\begin{array}{ccc} M & \xrightarrow{f} & \mathcal{F}_m(\mathbb{E}^N) \\ & \searrow u & \downarrow x \\ & & \mathbb{E}^N \end{array}$$

Step 4:

According to proposition 4.2.2., the existence of an integral manifold of \mathcal{I}_0 for wick Δ is non zero, is a neccessary condition for the existence of a local isometric embedding. However, the theorems and the results that we discussed deal with closed exterior differential system . Therefore it is natural to add to the Pffafian system I_0 the exterior differentiation of each 1-form. We obtain so a closed exterior differential system: $I_0 \cup dI_0$. When we compute the exterior differentiation of $(\omega_i - \eta_i)$, we remark new differential forms and an interesting result,

$$d(\omega_i - \eta_i) = - \sum_{j=1}^m (\omega_{ij} - \eta_{ij}) \wedge \omega_i = 0 \quad (23)$$

By Cartan's lemma, $\omega_{ij} - \eta_{ij} = \sum_{k=1}^m h_{ijk} \omega_k$, with $h_{ijk} = h_{ikj} = -h_{jik}$. With

the symmetry and the skew-symmetry of the functions h_{ijk} , we conclude that h_{ijk} are zero and so, $\omega_{ij} - \eta_{ij} = 0$. This result has a geometric intrepretation: $\omega_{ij} - \eta_{ij} = 0$ implies that $f^*(\omega_{ij}) = \eta_{ij}$ where f is the function of proposition. 4.2.2, which means that the pull-back of Lévi-Civita connection by an isometric embedding is the Lévi-Civita connection on M .

So, we extend the exterior differential I_0 and we obtain an exterior differential system on $M \times \mathcal{F}_m(\mathbb{E}^N)$ $I_1 = \{(\omega_i -$

⁸Conversely, each local isometric embedding $u : M \longrightarrow \mathbb{E}^N$ come uniquely from this construction.

$\eta_i)_{i=1,\dots,m}, (\omega_a)_{a=m+1,\dots,N}, (\omega_{ij} - \eta_{ij})_{1 \leq i < j \leq m}\}$. In order to have a closed one, we add the exterior differentiation of each form and we get $I = I_1 \cup dI_1$. We denote \mathcal{I} the exterior differential ideal generated by $I = \{(\omega_i - \eta_i), \omega_a, (\omega_{ij} - \eta_{ij}), d(\omega_i - \eta_i), d\omega_a, d(\omega_{ij} - \eta_{ij})\}$.

Instead of looking for integral manifold of \mathcal{I}_0 , we will look for the existence of an integral manifold of \mathcal{I} .

From the structure equations stated earlier, we obtain the following system:

$$\begin{cases} d(\omega_i - \eta_i) \equiv 0 & \text{mod } I_1 \\ d\omega_a \equiv -\sum_{i=1}^m \omega_{ai} \wedge \omega_i & \text{mod } I_1 \\ d(\omega_{ij} - \eta_{ij}) \equiv \sum_{a=m+1}^N \omega_{ai} \wedge \omega_{aj} - \Omega_{ij} & \text{mod } I_1 \end{cases} \quad (24)$$

On \mathbb{X} , the integral manifold of \mathbb{X} , $\omega_a = 0$, so $d\omega_a = 0$ too. We conclude that $\sum_{i=1}^m \omega_{ai} \wedge \omega_i = 0$. The Cartan lemma (lemma 1.5.1., page 300) ensures the

existence of m^2 functions h_{aij} such that $\omega_{ai} = \sum_{j=1}^m h_{aij} \omega_j$ where $h_{aij} = h_{aji}$.

We can write then: $\omega_{ai} - \sum_{j=1}^m h_{aij} \omega_j = 0$ on \mathbb{X} .

However, nothing lead us to think that this equality will be true outside \mathbb{X} . We define then the differential 1-form π_{ai} on $M \times \mathcal{F}_m(\mathbb{R}^N)$ as follows

$$\pi_{ai} = \omega_{ai} - \sum_{j=1}^m h_{aij} \omega_j \quad (25)$$

Consider now, the last equation of the system (24)

$$d(\omega_{ij} - \eta_{ij}) \equiv \sum_{a=m+1}^N \omega_{ai} \wedge \omega_{aj} - \Omega_{ij} \quad \text{mod } I \quad (26)$$

On \mathbb{X} , $\omega_{ij} - \eta_{ij} = 0$, so $d(\omega_{ij} - \eta_{ij}) = 0$. restricted to \mathbb{X} , (26) becomes

$$\sum_{a=m+1}^N \omega_{ai} \wedge \omega_{aj} = \Omega_{ij}. \quad (27)$$

Using (25), we can write (27) as follows

$$\text{On } \mathbb{X}: \quad \Omega_{ij} = \sum_{k,l=1}^m \left(\sum_{a=m+1}^N (h_{aik}h_{ajl} - h_{ail}h_{ajk}) \right) \omega_k \otimes \omega_l \quad (28)$$

from $\Omega_{ij} = \sum_{k,l=1}^m \mathcal{R}_{ijkl} \eta_k \otimes \eta_l = \sum_{k,l=1}^m \mathcal{R}_{ijkl} \omega_k \otimes \omega_l$, we conclude that

$$\sum_{a=m+1}^N (h_{aik}h_{ajl} - h_{ail}h_{ajk}) = \mathcal{R}_{ijkl} \quad (29)$$

Equation (29) is called the Gauss equation.

We see that the exterior differential system $\tilde{I} = \{(\omega_i - \eta_i), \omega_a, (\omega_{ij} - \eta_{ij}), \pi_{ai}\}$ when the Gauss's equation is satisfied, generates the exterior differential ideal \mathcal{I} . Actually, the 1-forms $(\omega_i - \eta_i)$ and ω_a belong to I and to \tilde{I} . The 1-forms $(\omega_{ij} - \eta_{ij}) = 0$. This implies that $d(\omega_i - \eta_i) = 0$. The 1-forms $\pi_{ai} = 0$, so $d\omega_a = 0$. From the Gauss equation, $d(\omega_{ij} - \eta_{ij}) = 0$. Looking for integral elements of I is equivalent to looking for integral elements of \tilde{I} for which the Gauss equation is satisfied. We shall proceed this in the following steps. Moreover, \tilde{I} contains less differential 1-form than the exterior differential system I .

Step 5:

The functions h_{aij} are symmetric in their two last indeces. If we consider an $(N - m)$ -dimensional euclidean space \mathcal{W} , then the matrix (h_{aij}) can be viewed as a symmetric element of $\mathbb{R}^m(i, j = 1, \dots, m)$ taking value in \mathcal{W} , i.e. $(h_{aij}) \in \mathcal{W} \otimes S^2(\mathbb{R}^m)$. Notice that $\dim \mathcal{W} \otimes S^2(\mathbb{R}^m) = (N - m) \frac{m(m+1)}{2}$.

Proposition 4.2.3. *Let \mathcal{K}_m the set of Riemannian curvature tensors \mathcal{R} such that:*

1. $\mathcal{R}_{ijkl} = \mathcal{R}_{klij}$.
2. $\mathcal{R}_{ijkl} = -\mathcal{R}_{jikl}$.
3. $\mathcal{R}_{ijkl} + \mathcal{R}_{kijl} + \mathcal{R}_{jkil} = 0$.

where the indeces i, j, k and l vary from 1 to m . Then

$$\dim \mathcal{K}_m = \frac{m^2(m^2 - 1)}{12} \quad (30)$$

Lemma 4.2.1. *Suppose that $r = N - m \geq \frac{m(m-1)}{2}$. Let $\mathcal{H} \subset \mathcal{W} \otimes S^2(\mathbb{R}^m)$ an open set containing the elements $h = (h_{ij})$ such that the vectors $\{h_{ij} | 1 \leq i \leq j \leq m-1\}$ are linearly independents as elements of \mathcal{W} . The map $\gamma : \mathcal{H} \rightarrow \mathcal{K}_m$ that for $h \in \mathcal{H}$ associate $\gamma(h) \in \mathcal{K}_m$ such that $(\gamma(h))_{ijkl} = \sum_{a=m+1}^N h_{aik}h_{ajl} - h_{ail}h_{ajk}$, is a surjective submersion.*

Step 6: The existence of an m -dimensional ordinary integral element

Let \mathcal{I} the exterior ideal of $M \times \mathcal{F}_m(\mathbb{E}^N)$ generated by $s = N(m+1) - \frac{m(m+1)}{2}$ 1-forms:

$$\left\{ \underbrace{(\omega_i - \eta_i)_{i=1, \dots, m}}_m, \underbrace{(\omega_a)_{a=m+1, \dots, N}}_{N-m}, \underbrace{(\omega_{ij} - \eta_{ij})_{1 \leq i < j \leq m}}_{\frac{m(m-1)}{2}}, \underbrace{(\pi_{ai})_{i=1, \dots, m, a=m+1, \dots, N}}_{(N-m)m} \right\}$$

Let $\mathcal{Z} = \{(x, \Upsilon, h) \in M \times \mathcal{F}_m(\mathbb{E}^N) \times \mathcal{H} | \gamma(h) = \mathcal{R}(x)\}$. \mathcal{Z} is a submanifold (the fiber of \mathcal{R} by a submersion. The surjectivity of γ ensure that $\mathcal{Z} \neq \emptyset$). So,

$$\begin{aligned} \dim \mathcal{Z} &= \dim M + \dim \mathcal{F}_m(\mathbb{E}^N) + \dim \mathcal{H} - \dim \mathcal{K}_m \\ &= \underbrace{m}_{\dim M} + \underbrace{N(m+1) - \frac{m(m+1)}{2}}_{\dim \mathcal{F}_m(\mathbb{E}^N)} + \underbrace{(N-m)\frac{m(m+1)}{2}}_{\dim \mathcal{H}} - \underbrace{\frac{m^2(m^2-1)}{12}}_{\dim \mathcal{K}_m} \end{aligned} \quad (31)$$

We define the map $\Phi : \mathcal{Z} \rightarrow \mathcal{V}_m(\mathcal{I}, \Delta)$ that associate to (x, Υ, h) the m -plane at (x, Υ) annihilated by the 1-forms that generate \mathcal{I} (the exterior differential system \tilde{I}). The map Φ is an embedding and so $\Phi(\mathcal{Z})$ is a submanifold of $\mathcal{V}_m(\mathcal{I}, \Delta)$. We will show that $\Phi(\mathcal{Z})$ contains only ordinary integral elements. In the proof, we will use the proposition.3.2.1.

Let $(x, \Upsilon, h) \in \mathcal{Z}$ be a point. Let denote $E = \Phi(x, \Upsilon, h)$ the integral element defined as follows: $E = \{v \in T_{(x, \Upsilon)}(M \times \mathcal{F}_m(\mathbb{E}^N)) | (\omega_i - \eta_i)(v) = \omega_a(v) = (\omega_{ij} - \eta_{ij})(v) = \pi_{ai}(v) = 0\}$.

E is an m -dimensional integral element. As a matter of fact, s is the number of differential forms that generate the ideal \mathcal{I} and

$$\dim(M \times \mathcal{F}_m(\mathbb{E}^N)) - m = N(m+1) - \frac{m(m+1)}{2} = s.$$

We will apply word by word the proposition 3.2.1. Let \mathcal{I} the exterior ideal of $M \times \mathcal{F}_m(\mathbb{E}^N)$ defined above⁹. This ideal does not contain 0-forms. $E \in \mathcal{V}_m(\mathcal{I})$ at $(x, \Upsilon) \in M \times \mathcal{F}_m(\mathbb{E}^N)$. Let $\omega_i, (\omega_i - \eta_i), \omega_a, (\omega_{ij} - \eta_{ij}), \pi_{ai}$ be a coframe¹⁰ of $M \times \mathcal{F}_m(\mathbb{E}^N)$ in the neighborhood of (x, Υ) such that¹¹ $E = \{v \in T_{x, \Upsilon}(M \times \mathcal{F}_m(\mathbb{E}^N)) | (\omega_i - \eta_i)(v) = \omega_a(v) = (\omega_{ij} - \eta_{ij})(v) = \pi_{ai}(v) = 0\}$.

For $p \leq m$, we define the p -dimensional integral element¹² $E_p = \{x \in E | \omega_k(v) = 0 \text{ pour tout } k > p\}$.¹³ We obtain so, an integral flag $(0)_{(x, \Upsilon)} = E_0 \subset E_1 \subset E_2 \subset \dots \subset E_m \subset T_{(x, \Upsilon)}(M \times \mathcal{F}_m(\mathbb{E}^N))$. We remind that $I = \underbrace{\{(\omega_i - \eta_i), \omega_a, (\omega_{ij} - \eta_{ij})\}}_{\text{differential 1-forms}}, \underbrace{d(\omega_i - \eta_i), d\omega_a, d(\omega_{ij} - \eta_{ij})}_{\text{differential 2-forms}}\}..$

By computing $d(\omega_i - \eta_j), d\omega_a$ and $d(\omega_{ij} - \eta_{ij})$, we shall find the differential forms that are linear combinations of the forms which generate \mathcal{I} .¹⁴

After simple calculations, we find that

$$d\omega_a \equiv - \sum_{i=1}^m \pi_{ai} \wedge \omega_i \quad (32)$$

and

$$d(\omega_{ij} - \eta_{ij}) = \underbrace{\sum_{a=m+1}^N \pi_{ai} \wedge \pi_{aj}}_{\diamond} + \sum_{k=1}^m \left(\sum_{a=m+1}^N h_{ajk} \pi_{ai} - h_{aik} \pi_{aj} \right) \wedge \omega_k \quad (33)$$

the term (\diamond) is quadratic in π_{ai} and vanishes on \mathbb{X} .¹⁵

⁹ $M \times \mathcal{F}_m(\mathbb{E}^N)$ play the role of the manifold "M" in the proposition. 3.2.1.

¹⁰ There is $m + s = \dim(M \times \mathcal{F}_m(\mathbb{E}^N))$ 1-forms.

¹¹ the $(\omega_i)_{i=1, \dots, m}$ play the role of " $\omega_1, \omega_2, \dots, \omega_n$ ". the $(\omega_i - \eta_i), \omega_a, (\omega_{ij} - \eta_{ij}), \pi_{ai}$ play the role of " π_s " in the proposition 3.2.1.

¹² the exterior differential system I play the role of " $\{\varphi_1, \varphi_2, \dots, \varphi_r\}$ " in the proposition 3.2.1.

¹³ $E_p \in \mathcal{V}_p(\mathcal{I}, \Delta)$ Because it is annihilated by $s + m - p$ differential 1-forms .

¹⁴ the forms that play the role of π_ρ^J in the proposition 3.2.1.

¹⁵ (\diamond) play the role of $\tilde{\varphi}_\rho$ in the proposition 3.2.1.

According to proposition 3.2.1., c_p represents the number of linear independent differential 1-forms.

The differential 1-forms	The indexes	Number of linear independent 1-forms
$\omega_i - \eta_i$	$1 \leq i \leq m$	m
ω_a	$m+1 \leq a \leq N$	$N - m$
$\omega_{ij} - \eta_{ij}$	$1 \leq i < j \leq m$	$\frac{m(m-1)}{2}$
π_{ai}	$1 \leq i \leq p,$ $m+1 \leq a \leq N$	$(N-m)p$
$\sum_{a=m+1}^N (h_{aik}\pi_{aj} - h_{ajk}\pi_{ai})$	$1 \leq k \leq p,$ $1 \leq i \leq j \leq m$	$p \frac{(m-p)(m-p-1)}{2} +$ $\frac{p(p+1)}{2}(m-p)$

Finally, by the sum of the number of linear independent 1-forms of the above table, c_p is the codimension of $H(E_p)$ in $G_m\left(T(M \times \mathcal{F}_m \mathbb{E}^N)\right)$ defined earlier, and is equal to:

$$c_p = N + \frac{m(m-1)}{2} + (N-m)p + \frac{mp(m-p)}{2} \quad (34)$$

so,

$$\sum_{p=0}^{m-1} c_p = \frac{Nm(m+1)}{2} + \frac{m^2(m^2-1)}{12}. \quad (35)$$

To apply the proposition 3.2.1 and show that E is an m -dimensional ordinary integral element, we need just to compute the codimension of $\Phi(\mathcal{Z})$ in $G_m(\mathcal{I}, \Delta)$.

Let \mathfrak{U} an open set of $\mathcal{F}_m(\mathbb{E}^N)$. So, $\dim(M \times \mathfrak{U}) = N(m-1) - \frac{m(m+1)}{2} + m$. We remind that if E is an n -dimensional euclidean space, then the space of all p -planes of E ($G_p(E)$), with $p < n$, is of $p(n-p)$ dimension. Let $(x, \Upsilon) \in M \times \mathfrak{U}$,

$$\begin{aligned}
\dim G_m(T(M \times \mathfrak{U})) &= \dim G_m(T_{(x, \Upsilon)}(M \times \mathfrak{U})) + \dim(M \times \mathfrak{U}) \\
&= m \left(\underbrace{N(m+1) - \frac{m(m+1)}{2} + m - m}_{\dim T_{(x, \Upsilon)}(M \times \mathfrak{U}) = \dim(M \times \mathfrak{U})} \right) + \underbrace{N(m+1) - \frac{m(m+1)}{2} + m}_{\dim(M \times \mathfrak{U})} \\
&= m \left(N(m+1) - \frac{m(m+1)}{2} \right) + N(m+1) - \frac{m(m+1)}{2} + m
\end{aligned}$$

Since that Φ is an embedding, $\dim \Phi(\mathcal{Z}) = \dim \mathcal{Z}$, so

$$\begin{aligned}
&\dim G_m(T(M \times \mathfrak{U})) - \dim \Phi(\mathcal{Z}) = \\
&\underbrace{Nm(m+1) - \frac{m^2(m+1)}{2} + N(m+1) - \frac{m(m+1)}{2} + m}_{\dim G_m(T(M \times \mathfrak{U}))} \\
&\quad - \underbrace{-m - N(m+1) + \frac{m(m+1)}{2} - \frac{Nm(m+1)}{2} + \frac{m^2(m+1)}{2} + \frac{m^2(m^2-1)}{12}}_{\dim \Phi(\mathcal{Z})} \\
&= \frac{Nm(m+1)}{2} + \frac{m^2(m^2-1)}{12}
\end{aligned}$$

We conclude that the codimension of $\Phi(\mathcal{Z})$ in $G_m(T(M \times \mathfrak{U}))$ is equal to $c_0 + c_1 + \dots + c_{m-1}$. By the Cartan's test, $E \in \mathcal{V}_m(\mathcal{I}, \Omega)$ is an ordinary integral element of \mathcal{I} . The Cartan-Kähler theorem (Corollary 3.3.1.) ensure the existence of an integral manifold \mathbb{X} passing through (x, Υ) and having E as a tangent space at (x, Υ) .

$E \in \mathcal{V}_m(\mathcal{I}, \Omega)$, In particular, $E \in \mathcal{V}_m(\mathcal{I}_0, \Omega)$. By the proposition 4.2.2. , there exists an isometric embedding of (M, g) in $(\mathbb{E}^N, \varepsilon_N)$.

5 Local Conformal Embedding Problem

Definition 5.0.1. *Let (M, g) and (N, h) be two real analytic Riemannian manifolds of dimension m and n respectively. Let f be a map from M to N . Then f is a conformal embedding from (M, g) to (N, h) if:*

1. f is a local diffeomorphism;

2. $f^*h = Sg$, where $S : M \rightarrow \mathbb{R}^+$ is a strictly positive function on M .

Theorem 5.1. (Jacobowitch-Moore [5])

If $\dim N = n \geq \frac{1}{2}m(m+1) - 1$, then each point $p \in M$ admit a neighborhood on M which can be conformally embedded in N .

Jacobowitch and Moore gave two different proofs of this result; one is based on Janet's method and the second on Cartan's method which is close to the proof of Burstin-Cartan-Janet-Schlaefli theorem that we gave.

Roughly speaking, we consider $\mathcal{F}(M) \times \mathcal{F}(N) \times \mathbb{R}^m \times \mathbb{R}_*^+$ and we look for integral manifolds of $I_0 = \{\omega_i - S\eta_i, \omega_a\}$ (the forms are defined as on the previous section). Similarly, we can extend the exterior differential system to obtain a closed one. We lead the details of the proof for the reader who should take care of S when he apply the exterior differentiation cause it's a function. When the new exterior differential system is involutive, we look for ordinary integral element and so conclude. (see [5]).

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